

## From lattice to continuum percolation via clustering

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1990 J. Phys. A: Math. Gen. 23 L197

(<http://iopscience.iop.org/0305-4470/23/5/003>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 01/06/2010 at 09:59

Please note that [terms and conditions apply](#).

## LETTER TO THE EDITOR

# From lattice to continuum percolation via clustering

J W Evans

Ames Laboratory and Departments of Mathematics and Physics, Iowa State University,  
Ames, IA 50011, USA

Received 23 November 1989

**Abstract.** We consider the growth of Eden clusters on a square lattice about randomly distributed seeds, or through continuous nucleation. We analyse the dependence of the percolation threshold,  $p_c$ , on the seed density, or ratio of nucleation to growth rates, i.e. on parameters controlling the average size of individual clusters or islands before coalescence. We find that  $p_c$  approaches a large island size continuum limit from below at a rate depending on the active zone width scaling exponent. This result applies to a variety of island-forming models.

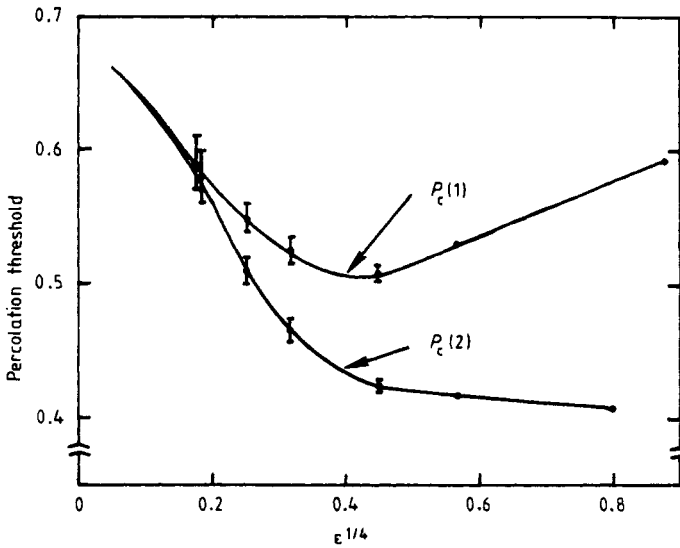
Several studies have confirmed that various lattice and continuum percolation problems are in the same universality class [1]. Domb had noted much earlier that continuum percolation of equal-sized circles or spheres can be obtained from lattice percolation in the limit of long-range connectivity [2]. More concisely, one assigns filled sites to the same cluster if they are separated by a Euclidean distance of  $R$  or less, and then lets  $R \rightarrow \infty$ . Changing from Euclidean to other metrics generates continuum percolation problems for equal-sized aligned objects of various shapes, as  $R \rightarrow \infty$ .

In this letter, we analyse another less trivial transition from lattice to continuum percolation associated with non-equilibrium models on a square lattice, wherein compact islands grow either about randomly distributed seeds or via continuous nucleation. These islands eventually coalesce to form percolating clusters. The continuum (large island size) limit is achieved as the seed density,  $\epsilon$ , or ratio of nucleation to growth rates,  $\alpha^{-1}$ , vanish. Specifically we focus on the case where islands are Eden clusters [3]. Thus for growth about seeds, all empty perimeter sites fill at the same rate, i.e. with equal probability. For continuous nucleation, empty sites with all neighbours empty fill with rate 1, and empty perimeter sites fill at rate  $\alpha \geq 1$ . Pictures of island distributions for these models, and their continuum limits, can be found in [4, 5].

The finite size scaling procedure used to estimate percolation thresholds,  $p_c$ , for these models is described in detail elsewhere [4, 6]. Briefly, for an  $L \times L$  square lattice with periodic boundary conditions, we determine the second moment of the number of sites per cluster,  $M_L(p)$ , as a function of occupancy,  $p$ . The  $p$ -values where the functions  $R_L \equiv M_{2L}/M_L$  intersect, for different  $L$ , should approach  $p_c$  as  $L \rightarrow \infty$ .  $R_L$ -values at the intersection points should approach  $2^{2+\gamma/\nu}$  where  $\gamma(\nu)$  is the cluster size (connectivity length) exponent. Since these models have finite-range correlations, one expects (and our results suggest) random percolation universality where  $2^{2+\gamma/\nu} \approx 13.85$ . Thus we can alternatively estimate  $p_c$  from the  $L \rightarrow \infty$  behaviour of  $p$ -values where  $R_L = 13.85$ . For growth about seeds with various  $\epsilon$ , such  $p$ -values are shown in table 1, and behaviour of the associated  $p_c$ -estimates is shown in figure 1. Corresponding

**Table 1.** Growth of Eden clusters about seeds of density  $\epsilon$ ;  $p$ -values are reported at which  $R_L = 13.85$  for filled clusters with 2nd NN [NN] connectivity. Uncertainties in the last digit are shown in parentheses ( ).

$\epsilon$	$L = 16$	$L = 32$	$L = 64$	$L = 128$	
0.1	0.421 [0.524]	0.417 [0.527]	0.419 [0.532]		(1)
0.04	0.442	0.430 [0.504]	0.424 [0.508]		(2)
0.01			0.473 [0.509]	0.464 [0.524]	(2)
0.004			0.526 [0.523]	0.513 [0.541]	(3)
0.001			[0.512]	0.605 [0.567]	(8)



**Figure 1.** Filled site percolation thresholds for NN connectivity,  $p_c(1)$ , and for 2nd NN connectivity,  $p_c(2)$ , for growth of Eden clusters about randomly distributed seeds of density,  $\epsilon$ .

**Table 2.** Continuous nucleation (at rate 1) and growth (at rate  $\alpha$ ) of Eden clusters;  $p$ -values are reported at which  $R_L = 13.85$  for filled clusters with 2nd NN [NN] connectivity. Uncertainties are also shown.

$\alpha$	$L = 32$	$L = 64$	$L = 128$	
49	0.448 [0.517]	0.438 [0.524]	0.441 [0.515]	(2)
150	0.475 [0.509]	0.468 [0.523]	0.459 [0.520]	(2)
499		0.502 [0.530]	0.490 [0.537]	(2)
2 000		[0.541]	0.540 [0.556]	(3)
15 000			0.630 [0.59]	( $\frac{1}{2}$ )

results for continuous nucleation with various  $\alpha$  are shown in table 2 and figure 2. Also included in figure 2 are previous  $p_c$ -estimates for continuous nucleation for a restricted range of 'small'  $\alpha$  [4, 5]. Results become less accurate as  $\varepsilon$  decreases and  $\alpha$  increases because of the larger statistical fluctuations associated with larger islands, and the need to use larger lattices to achieve convergence.

Early Ising model studies (for a review see [7]) fostered the view that the introduction of clustering (positive correlations) enhances percolation, i.e. reduces the filled site percolation threshold,  $p_c$ . However, more extensive numerical studies for both lattice and continuum systems reveal that clustering can either enhance or inhibit percolation depending on the choice of connectivity [8]. For a simple demonstration of this behaviour, consider the Ising model on a square lattice with nearest-neighbour ( $NN$ ) attractive interactions. For  $NN$  connectivity,  $p_c = p_c(1)$  decreases with increasing interaction strength [7, 8]. Then using particle-hole symmetry, the empty site percolation threshold for  $NN$  connectivity,  $p'_c(1) = 1 - p_c(1)$ , increases with increasing interaction strength. However,  $p'_c(1)$  must coincide with  $p_c(2)$ , the filled site percolation threshold for 2nd  $NN$  connectivity (cf [9]), since either the  $NN$ -empty or the 2nd  $NN$ -filled clusters must percolate (but not both since they cannot cross each other). Thus  $p_c(1)$  and  $p_c(2)$  are complementary and exhibit opposite behaviour with the introduction of clustering. Clearly this also holds for an equilibrated lattice gas with arbitrary-range *pairwise* interactions. Finally we note that for very general distributions of filled sites with clusters defined by 'long-range' connectivity, the introduction of 'short-range' clustering will clearly inhibit percolation, i.e. raise the threshold. Here percolation is enhanced for suitably dispersed, not clustered, distributions of filled sites.

Such behaviour of the filled site percolation threshold,  $p_c$ , is seen in our non-equilibrium models (figures 1 and 2). The introduction of clustering (i.e. by lowering  $\varepsilon$  below the independent site percolation threshold of 0.593, or by raising  $\alpha$  above unity) initially lowers  $p_c = p_c(1)$  for  $NN$  connectivity, but raises  $p_c$  for 2nd and longer-range connectivity. It should be noted that here there is no particle-hole symmetry, so opposite behaviour of  $p_c(1)$  and  $p_c(2)$  is not guaranteed. In this contribution, our

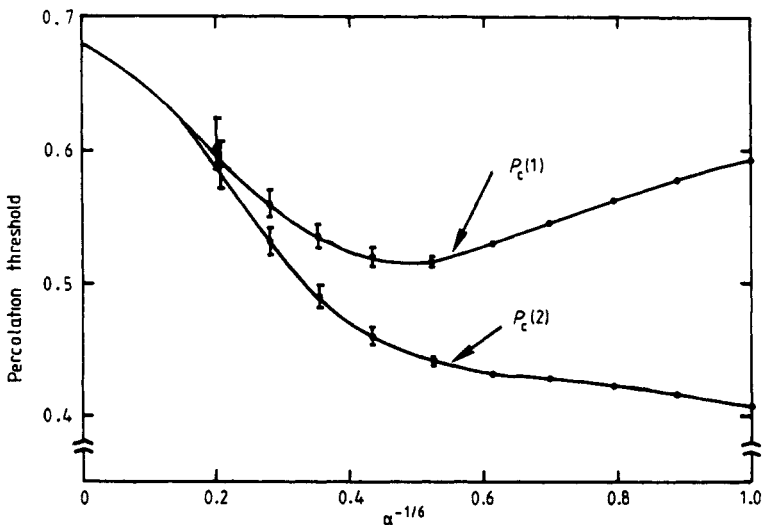


Figure 2. Filled site percolation thresholds,  $p_c(1)$  and  $p_c(2)$ , for continuous nucleation (at rate 1) and growth (at rate  $\alpha$ ) of Eden clusters.

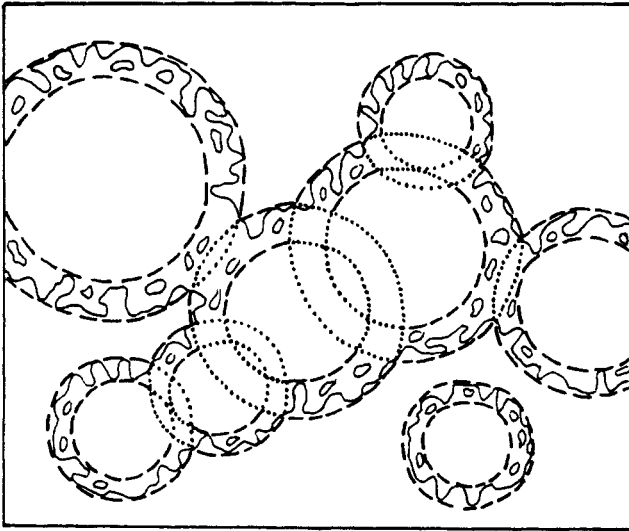
focus is on characterising the nature of the approach of  $p_c$  to its continuum limit. Thus the remaining discussion develops a simple picture (sketched previously [4]) of island and near-percolation cluster structure in the  $\varepsilon \ll 1$  or  $\alpha \gg 1$  large island regime.

An individual island or Eden cluster or characteristic radius  $R$  becomes asymptotically near-circular as  $R \rightarrow \infty$  [10], and has an active or growing zone at its perimeter of width  $\xi \sim R^{q(R)}$ . Here  $q(R)$  is an 'effective exponent' which varies from  $q \sim 0.52$  when  $R \sim 10$ , to  $q \sim 0.36$  when  $R \sim 10^2$ , to 0.50 for  $R \geq 10^3$  [11]. With each island, one naturally associates outer and inner convex (roughly circular) envelopes to the active zone. We anticipate that only the outer envelopes need link or overlap to form percolating clusters (see figure 3). Let  $\langle \rangle$  denote an average over individual islands. Then visualisation of the approach to the continuum limit, and analysis of  $p_c$ -dependence on  $\langle R \rangle$  and  $\langle \xi \rangle$ , is naturally facilitated by a length rescaling wherein the rescaled average island radius is unity, and the rescaled average active zone width is  $\langle \xi / R \rangle \sim \langle R \rangle^{q-1}$ . Thus the continuum limit,  $\langle R \rangle \rightarrow \infty$ , corresponds to solid islands with rescaled active zone widths of zero. Increasing the rescaled average width,  $\langle \xi / R \rangle$ , clearly reduces the fractional coverage or occupancy at percolation, i.e. fuzzy island edges enhance percolation. Since the islands are solid interior to the active zone (in contrast to diffusion-limited-aggregation clusters), it is reasonable to expect that

$$p_c(0) - p_c(\langle \xi / R \rangle) \sim O(\langle \xi / R \rangle) \sim O(\langle R \rangle^{q-1}) \quad \text{as } \langle R \rangle \rightarrow \infty$$

where  $p_c(0)$  represents the continuum limit of the threshold. This large- $\langle R \rangle$  scaling, and obviously  $p_c(0)$ , are independent of the choice of finite-range connectivity. This explains the coalescence of  $p_c(1)$  and  $p_c(2)$  curves in figures 1 and 2. Since  $p_c(2)$  corresponds to the percolation threshold for NN-empty clusters, the coalescence of these curves can also be explained by the necessary coincidence of continuum percolation thresholds for empty and filled regions [4].

We now apply this result to the models under consideration. For growth about seeds, clearly one has that  $\langle R \rangle \sim \varepsilon^{-1/2}$ , and consequently  $p_c$  approaches  $p_c(0)$  like



**Figure 3.** Schematic of the structure of continuously nucleated islands (e.g. Eden clusters), and a percolating cluster of such islands. The full lines represent the boundaries of filled regions. Outer and inner envelopes of active zones are indicated by broken lines.

$\varepsilon^{(1-q)/2} \sim \varepsilon^{1/4}$ , as  $\varepsilon \rightarrow 0$  (see figure 1). Here the continuum limit corresponds to the growth of near-circular islands, at constant rate, about randomly distributed points in the plane. Thus  $p_c(0)$  should be very close to the percolation threshold,  $0.677 \pm 0.004$  [1, 12] for equal sized discs randomly distributed in the plane. For continuous nucleation, one has that  $\langle R \rangle \sim \alpha^{1/3}$  [5, 13], so  $p_c$  approaches  $p_c(0)$  like  $\alpha^{-(1-q)/3} \sim \alpha^{-1/6}$ , as  $\alpha \rightarrow \infty$  (see figure 2). As noted previously [4], here the continuum limit corresponds to a new Johnson-Mehl [14] type percolation problem, where near-circular islands continuously nucleate and grow in the plane. From figure 2, we estimate its percolation threshold at about 0.7. This result is not surprising since continuum percolation thresholds seem fairly stable against introducing a 'regular' distribution of island radii [15]. (However, such  $p_c$  can approach unity for pathological distributions [16].)

In summary, we have developed some generic ideas for elucidating the influence of clustering on the percolation threshold and, in particular, the nature of the approach to the large island size continuum limit. The influence on  $p_c$  of introducing clustering clearly depends on the connectivity rule (cf [8]), and for non-equilibrium models of the type considered here we expect  $p_c$  to approach its continuum limit from below. The rate of convergence will be faster (slower) for compact islands with narrower (broader) active zones, i.e. smaller (larger)  $q \geq 0$ . For example, convergence will be much faster if one chooses multiplicative [4] (rather than Eden) rates, since then active zone widths are  $O(1)$  up to coalescence [4].

The author would like to acknowledge D E Sanders' contributions in developing the computer code. Ames Laboratory is operated for the US Department of Energy by Iowa State University under contract No W-7405-Eng-82. This work was supported by the Applied Mathematical Sciences subprogram of the Office of Basic Energy Sciences.

## References

- [1] Vicesk T and Kertesz J 1981 *J. Phys. A: Math. Gen.* **10** L31  
Gawlinski E T and Stanley H E 1981 *J. Phys. A: Math. Gen.* **14** L291
- [2] Domb C 1972 *Biometrika* **59** 209
- [3] Eden M 1960 *Proc. 4th Berkeley Symp. on Mathematics, Statistics and Probability* vol IV, ed F Neyman (Berkeley, CA: University of California Press) p 233
- [4] Evans J W and Sanders D E 1988 *J. Vac. Sci. Technol. A* **6** 726  
Sanders D E and Evans J W 1988 *Phys. Rev. A* **38** 4186
- [5] Anderson S R and Family F 1988 *Phys. Rev. A* **38** 4198
- [6] Saleur H and Derrida B 1985 *J. Physique* **46** 1043
- [7] Stauffer D, Coniglio A and Adam A 1982 *Adv. Polymer Sci.* **44** 103
- [8] Bug A L R, Safran S A, Grest G S and Webman I 1985 *Phys. Rev. Lett.* **55** 1896
- [9] Sykes M F and Essam J W 1964 *J. Math. Phys.* **8** 1117
- [10] Wolf D E 1987 *J. Phys. A: Math. Gen.* **20** 1251
- [11] Freche P, Stauffer D and Stanley H E 1985 *J. Phys. A: Math. Gen.* **18** L1163
- [12] McCarthy J F 1987 *J. Phys. A: Math. Gen.* **20** 6053
- [13] Evans J W, Bartz J A and Sanders D E 1986 *Phys. Rev. A* **34** 1434
- [14] Johnson W A and Mehl R F 1939 *Trans. AIME* **135** 416
- [15] Kertesz J and Vicsek T 1982 *Z. Phys. B* **45** 345
- [16] Phani M K and Dhar D 1984 *J. Phys. A: Math. Gen.* **17** L645